Nonlinear Vibrations of Unsymmetrically Laminated Beams

Rakesh K. Kapania* and Stefano Raciti†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

The purpose of this study was to develop a simple one-dimensional finite element for the nonlinear analysis of symmetrically and unsymmetrically laminated composite beams including shear deformation. The beam element has 10 degrees of freedom at each of the two nodes: the axial displacement, the transverse deflection and the slope due to bending and shear, the twisting angle, the in-plane shear rotation, and their derivatives. The formulation, the solution procedure, and the computer program have been evaluated by solving a series of examples on the static response, free vibration, and nonlinear vibrations of isotropic and laminated beams. For unsymmetrically laminated beams, the nonlinear vibrations were found to have a soft spring behavior for certain boundary conditions as opposed to a hard spring behavior observed in isotropic and symmetrically laminated beams. The in-plane boundary conditions were found to have a significant effect on nonlinear responses.

Introduction

DURING the past two decades, composite materials have become widely used in the design of structural elements. The aerospace industry can be considered the promoter for the advancement of structural design through the use of new advanced composite materials. The greatest advantages of these materials are their high stiffness-to-weight ratio and high strength-to-weight ratio, which have a great potential for reducing structural weight.

A considerable number of studies on composite beams and plates have shown the importance of shear deformation in the analysis of these structural components. The transverse shear moduli of composite materials are usually very low compared to the in-plane tensile moduli, with the result that transverse shear deformation can be of considerable importance compared to homogeneous isotropic materials. A beam theory that includes shear deformation effects is generally referred to as Timoshenko's beam theory.

Methods for including shear deformation in the analysis of plates are given, among others, by Fricker,[1] Di Sciuva,[2] Whitney and Pagno,[3] Reddy and Kuppusamy,[4] Bhashyam and Gallagher,[5] Phan and Reddy,[6] and Reddy.[7] The importance of shear deformation in laminated beams is explained in the papers by, among others, Chen and Yang,[8] Teh and Huang,[9] Whitney et al.,[10] Teoh and Huang,[11] Murty and Shimpi,[12] Miller and Adams,[13] Abarcar and Cunniff,[14] and Hu et al.[15] All of these papers are restricted to the analysis of symmetrically laminated beams.

In laminated composite beams, the transverse shear deformation significantly affects the lateral displacement, the natural frequencies of vibration, and the buckling loads. The classical beam theory, which is based on the Bernoulli-Euler assumption that planes initially normal to the midplane remain plane and normal to the midsurface after bending, leads to high percentages of error in the analysis of beams or plates composed of anisotropic materials. For example, the classical plate theory predicts natural frequencies 25% higher than those predicted by a plate shear deformation theory for plates with side-to-thickness ratio of 10.[16] Also, results from Ref. 4 show that the first-order shear deformation theory discussed by Reddy and Chao,[17] predicts values of the natural frequencies that are 11% higher than those predicted by three-dimensional elasticity theory. A similar behavior is predicted to characterize beam vibrations when the transverse shear deformations are neglected. Furthermore, a beam element including shear effects is also important for studying random vibrations[18] and wave propagation.[19]

A study on the free vibration of fiber-reinforced cantilever beams is given in Ref. 14. The authors included shear deformations in their analysis by splitting the slope of the deflection curve into two parts. One part contains the rotation due to direct bending and induced bending due to twisting, and the other contains the shearing angle of distortion due to the application of a shear force. A slightly different approach was used by Murty and Shimpi,[12] who included shear deformations by dividing the total transverse displacement into bending displacement plus the transverse shear deflection. The research done by Teoh and Huang[11] on the free vibration analysis of fiber-reinforced beams clearly shows the effect of shear deformation on the higher bending modes of vibration, whereas the twisting mode is practically unaffected.

In order to take into account the parabolic through-the-thickness distribution of shear stresses, Timoshenko's beam theory uses a shear correction factor k. The most widely accepted definition is that k represents the ratio of the average shear strain on a section to the shear strain at the centroid.[20] There has been criticism of this definition of k. Thus, other formulations were developed, as given in Refs. 20 and 21. A closed-form solution for the shear factor of laminated nonhomogeneous materials is given by Bert[22] and Bert and Gordaninejad.[23] Both studies use a simple mechanics-of-materials approach, in which the shear strain energy of a nonhomogeneous beam is equated to the shear strain energy of an equivalent Timoshenko beam with uniform transverse shear stress distribution.

During recent years, the nonlinear analysis of beams has gained considerable attention for increasing the efficiency of structural design. A survey on the theoretical and computational advances in the nonlinear analysis of beams has been given by Sathyamoorthy.[24] More recently, the finite-element method has become widely used in the nonlinear analysis of structural elements as a relatively simple and reliable approach for solving a large variety of problems. As a result, numerous finite-element programs, for both the material and geometric nonlinearities for the static and dynamic behavior of structural elements, have been developed. A review of the extensive

Presented as Paper 87-0859 at 28th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics and Materials Conference, Monterey, CA, April 6–8, 1987; received June 1, 1987; revision received Jan. 4, 1988.

*Assistant Professor, Department of Aerospace and Ocean Engineering. Member AIAA.
†Research Assistant.

work carried out in the nonlinear analysis of beams using finite-element methods is given by Sathyamoorthy.[25]

This study was motivated by the lack of a simple one-dimensional finite-element analysis of unsymmetrically laminated beams including shear deformation, which could perform static, free-vibration, buckling, and nonlinear vibrations analysis. A recent finite-element formulation for the linear analysis of symmetrically laminated beams was presented by Chen and Yang,[8] where a 12-DOF element was used. This research differs from that of Chen and Yang in that an element with 20 DOF is developed and used. The element also allows for the geometrically nonlinear analysis (nonlinear vibrations) of unsymmetrically laminated beams. The purpose of this research is to develop a computer program to perform a nonlinear analysis of laminated composite beams in a simple and effective manner.

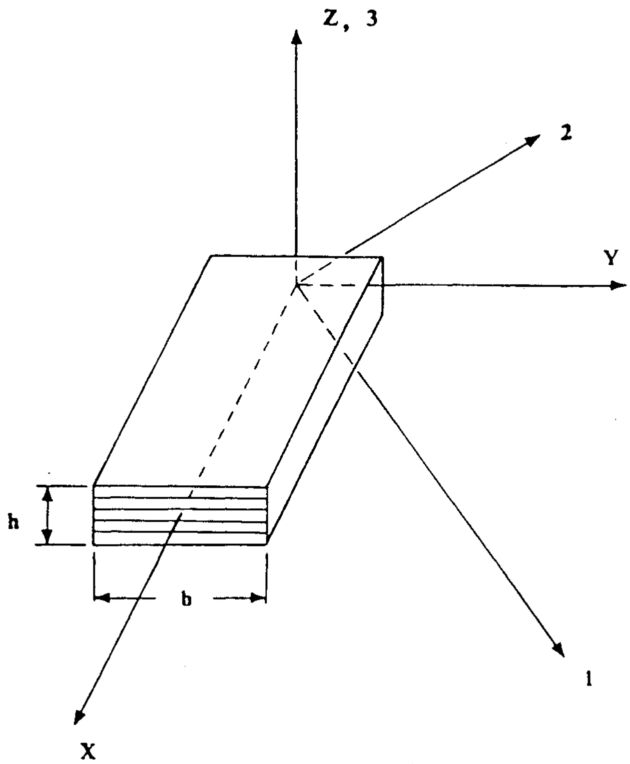

Fig. 1 Geometry of the beam element.

Element Formulation

The beam element is made of layers of orthotropic material in which the orthotropic axes of layer may be oriented at an arbitrary angle with respect to the beam axis. A simple one-dimensional beam element with 20 DOF is used to perform static, free-vibration, and nonlinear vibration analysis using a simple, efficient, and general approach.

Beam Element

The geometry of the one-dimensional beam element is shown in Fig. 1, and the 20 DOF are described in Fig. 2. The element has 10 DOF at each of the two nodes; the axial displacement u, the deflection due to bending w_b, the deflection due to shear w_s, the twisting angle τ, the in-plane shear $\beta(=du/dy)$, and their derivatives with respect to x. Both τ and β are assumed to be constant along the width (y axis) of the beam. Two coordinate systems are used: the x-y Cartesian coordinates and the ξ local coordinate system. The latter coordinate system, which ranges from -1 at node 1 to 1 at node 2, is used to perform the Gaussian numerical integration and will be used throughout this formulation.

Displacement Functions

The deflection behavior of the beam element is described by the displacement functions $u(\xi)$, $\beta(\xi)$, $w_b(\xi)$, $w_s(\xi)$, and $\tau(\xi)$ in terms of the nodal displacements that are interpolated through Hermitian polynomials as follows:

$$u(\xi) = N_1u_1 + N_2u_1' + N_3u_2 + N_4u_2' \tag{1}$$

where the prime stands for $d/d\xi$, and the N are the well-known Hermitian polynomials. Similar expressions are employed for $\beta(\xi)$, $w_b(\xi)$, $w_s(\xi)$, and $\tau(\xi)$.

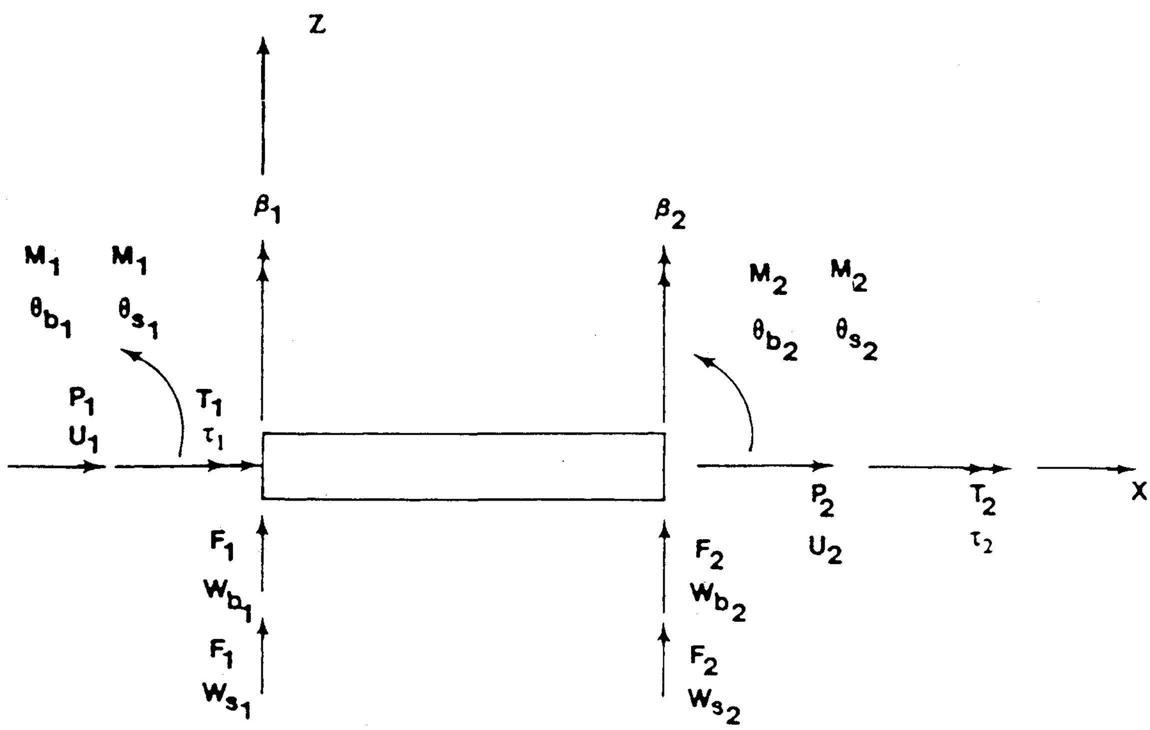

Fig. 2 20-degrees-of-freedom beam element.

Laminate Constitutive Relations

The mechanical response of the laminated beam under applied loads or external excitations is predicted in terms of the properties of the lamina by using classical lamination theory (CLT). The basic constitutive relations of the CLT are in the form

$$\begin{Bmatrix} N \\ \hline M \end{Bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B & D \end{array}\right] \begin{Bmatrix} \epsilon \\ \hline k \end{Bmatrix} \tag{2}$$

where the N are the resultant forces (forces per unit length), M the resultant moments (moments per unit length), the ϵ strains due to in-plane forces, and k the bending and twisting curvatures; the $[A]$, $[B]$, and $[D]$ matrices are, respectively, the extensional, coupling, and bending stiffness matrices.

For the case of the beam, the N_y resultant force and the M_y bending moment, from Eq. (2), can be equated to zero as follows:

$$N_y = A_{12}\epsilon_x + A_{22}\epsilon_y + A_{26}\gamma_{xy} + B_{12}k_x + B_{22}k_y + B_{26}k_{xy} = 0 \tag{3}$$

$$M_y = B_{12}\epsilon_x + B_{22}\epsilon_y + B_{26}\gamma_{xy} + D_{12}k_x + D_{22}k_y + D_{26}k_{xy} = 0 \tag{4}$$

However, the in-plane strain ϵ_y and the bending curvature k_y are assumed nonzero. Solving Eqs. (3) and (4) simultaneously for ϵ_y and k_y, the following is obtained:

$$\epsilon_y = \left[\left(A_{12} - \frac{B_{22}B_{12}}{D_{22}}\right)\epsilon_x + \left(A_{26} - \frac{B_{26}B_{22}}{D_{22}}\right)\gamma_{xy} + \left(B_{12} - \frac{D_{12}B_{22}}{D_{22}}\right)k_x + \left(B_{26} - \frac{D_{26}B_{22}}{D_{22}}\right)k_{xy}\right] \Bigg/ \left(\frac{B_{22}^2}{D_{22}} - A_{22}\right)$$

$$k_y = \left[\left(B_{12} - \frac{B_{22}A_{12}}{A_{22}}\right)\epsilon_x + \left(B_{26} - \frac{A_{26}B_{22}}{A_{22}}\right)\gamma_{xy} + \left(D_{12} - \frac{B_{12}B_{22}}{A_{22}}\right)k_x + \left(D_{26} - \frac{B_{26}B_{22}}{A_{22}}\right)k_{xy}\right] \Bigg/ \left(\frac{B_{22}^2}{A_{22}} - D_{22}\right) \tag{5}$$

Setting N_y and M_y equal to zero in Eq. (2) and adding the contribution due to ϵ_y and k_y, the constitutive equations become

$$\begin{Bmatrix} N_x \\ N_{xy} \\ M_x \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{16} & B_{11} & B_{16} \\ A_{16} & A_{66} & B_{16} & B_{66} \\ B_{11} & B_{16} & D_{11} & D_{16} \\ B_{16} & B_{66} & D_{16} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \gamma_{xy} \\ k_x \\ k_{xy} \end{Bmatrix} + \begin{bmatrix} A_{12} & B_{12} \\ A_{26} & B_{26} \\ B_{12} & D_{12} \\ B_{26} & D_{26} \end{bmatrix} \begin{Bmatrix} \epsilon_y \\ k_y \end{Bmatrix} \tag{6}$$

where ϵ_y and k_y are given by Eq. (5). The constitutive relations are symbolically written in the form

$$\{N\} = [D]\{\epsilon\} \tag{7}$$

Shear deformation is included in this formulation by splitting the transverse displacement into two parts:

$$w = w_b + w_s \tag{8}$$

where w is the total transverse displacement and w_b and w_s are as defined before, and by including the shear coefficient k in the transverse shear force-strain relation,

$$Q_x = kD_{44}\gamma_x \tag{9}$$

where Q_x is the transverse shear force, k the shear coefficient, D_{44} the transverse shear stiffness, and γ_x the transverse shear strain.

Strain-Displacement Relations

The strain relations are written in terms of displacement derivatives in the following form:

$$\epsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w_b}{\mathrm{d}x}\right)^2, \qquad \gamma_{xy} = \beta$$

$$k_x = \frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2}, \qquad k_{xy} = 2\frac{\mathrm{d}\tau}{\mathrm{d}x}, \qquad \gamma_x = \frac{\mathrm{d}w_s}{\mathrm{d}x} \tag{10}$$

Substituting the displacement equation (1) into the strain-displacement relations (10), the symbolic form of the strain-displacement relations may be written as

$$\{\epsilon\} = [B_0 + \tfrac{1}{2}B_L(q_i)]\{q_i\} \tag{11}$$

where $\{q_i\}$ is the nodal displacement vector, $[B_0]$ is the linear strain-displacement matrix, and $\tfrac{1}{2}[B_L(q_i)]$ is the nonlinear strain-displacement vector, as described in the following section.

Element Elastic Stiffness Matrix

The strain energy expression for the beam element is given as

$$U = \frac{1}{2}\int_0^L b\{\epsilon\}^T\{N\}\,\mathrm{d}x \tag{12}$$

where b is the width of the beam, $\{\epsilon\}$ the strain-curvature vector defined by Eq. (11), and $\{N\}$ the force and moment resultants vector. Substituting Eqs. (7) and (11) into Eq. (12), the strain energy relation becomes

$$U = \tfrac{1}{2}\{q\}^T\left[\int_0^L b[B]^T[D][B]\,\mathrm{d}x\right]\{q\} \tag{13}$$

where $[B]$, the total strain-displacement matrix, is given as

$$[B] = [B_0] + \tfrac{1}{2}[B_L(q_b)] \tag{14}$$

In the preceding expression, $[B_L(q_b)]$ is the nonlinear strain-displacement matrix and is given as

$$[B_L(q_b)] = [A][G] \tag{15}$$

where the $[A]$ and $[G]$ matrices are obtained as follows:

$$\epsilon_L = \tfrac{1}{2}\begin{bmatrix} \frac{\partial w_b}{\partial x} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial w_b}{\partial x} \end{Bmatrix} = \tfrac{1}{2}[A]\{\theta\} \tag{16}$$

The derivative of w_b can be related to the nodal displacements by

$$\frac{\partial w_b}{\partial x} = \frac{2}{L}(N_1' w_{b_1} + N_2'\theta_{b_1} + N_3' w_{b_2} + N_4'\theta_{b_2}) \tag{17}$$

or in matrix form as

$$\{\theta\} = \begin{Bmatrix} \frac{\partial w_b}{\partial x} \end{Bmatrix} = [G]\{q_b\} \tag{18}$$

where the superscript b refers to the transverse displacements (w_b and θ_b) due to bending only. Substituting Eq. (14) in Eq. (13), the strain energy relation becomes

$$U = \tfrac{1}{2}\{q\}^T\left[\int_0^L b[B_0 + \tfrac{1}{2}B_L(q_b)]^T[D] \times [B_0 + \tfrac{1}{2}B_L(q_b)]\,\mathrm{d}x\right]\{q\} \tag{19}$$

from which the total stiffness matrix $\bar{k}$ is obtained by

$$\bar{k} = \frac{\partial U}{\partial q_i} = \int_0^L b\Big([B_0]^T[D][B_0]\Big)\,\mathrm{d}x + \int_0^L b\Big([B_L^T(q_b)][D][B_0] + \tfrac{1}{2}[B_0]^T[D][B_L(q_b)]\Big)\,\mathrm{d}x + \frac{1}{2}\int_0^L b[B_L^T(q_b)][D][B_L(q_b)]\,\mathrm{d}x$$

$$i = 1,2,\ldots,20 \tag{20}$$

or symbolically written as

$$[\bar{k}] = [k] + [n_1] + [n_2] \tag{21}$$

where $[n_1,(q_b)]$ and $[n_2(q_b)]$ are the first- and second-order stiffness matrices, respectively. That is, $[n_1(q_b)]$ depends linearly and $[n_2(q_b)]$ depends quadratically on the element DOF due to bending, namely, w_{b_1}, θ_{b_1}, w_{b_2}, and θ_{b_2}.

The integrations given in Eq. (20) are performed using numerical integration by means of the Gaussian quadrature as follows:

$$I = \int_{-1}^{1} f(\xi)\,\mathrm{d}\xi = \sum_{i=1}^{n} H_i f(\xi_i) \tag{22}$$

which is the general form of the Gaussian quadrature, and where n is the number of function evaluations, H_i the weights, and ξ_i the abscissas of the sampling points.

Equations of Motion

The equations of motion can be derived using the Lagrange equations of motion, which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial U}{\partial q_i} = 0 \tag{23}$$

where U is the strain energy, and T the kinetic energy, is given as

$$T = \frac{1}{2}\int_0^L \rho A[\dot{w}_s(x) + \dot{w}_s(x)]^2\,\mathrm{d}x + \frac{1}{2}\int_0^L J\dot{\tau}^2(x)\,\mathrm{d}x \tag{24}$$

where A is the cross-sectional area of the beam, $J = \rho h b^3/12$ is the polar mass moment of inertia, and the (˙) stands for (d/dt). Substituting Eqs. (24) and (19) into Lagrange's equation (23), the equations of motion for nonlinear vibrations may be written as

$$[[k] + [n_1(q_b)] + [n_2(q_b)]]\{q\} + [m]\{\ddot{q}\} = \{0\} \tag{25}$$

where $[m]$ is the mass matrix, and the other terms are as defined earlier. Partitioning the $[k]$, $[n_1(q_b)]$, $[n_2(q_b)]$, and $[m]$ matrices so that the transverse displacements (due to

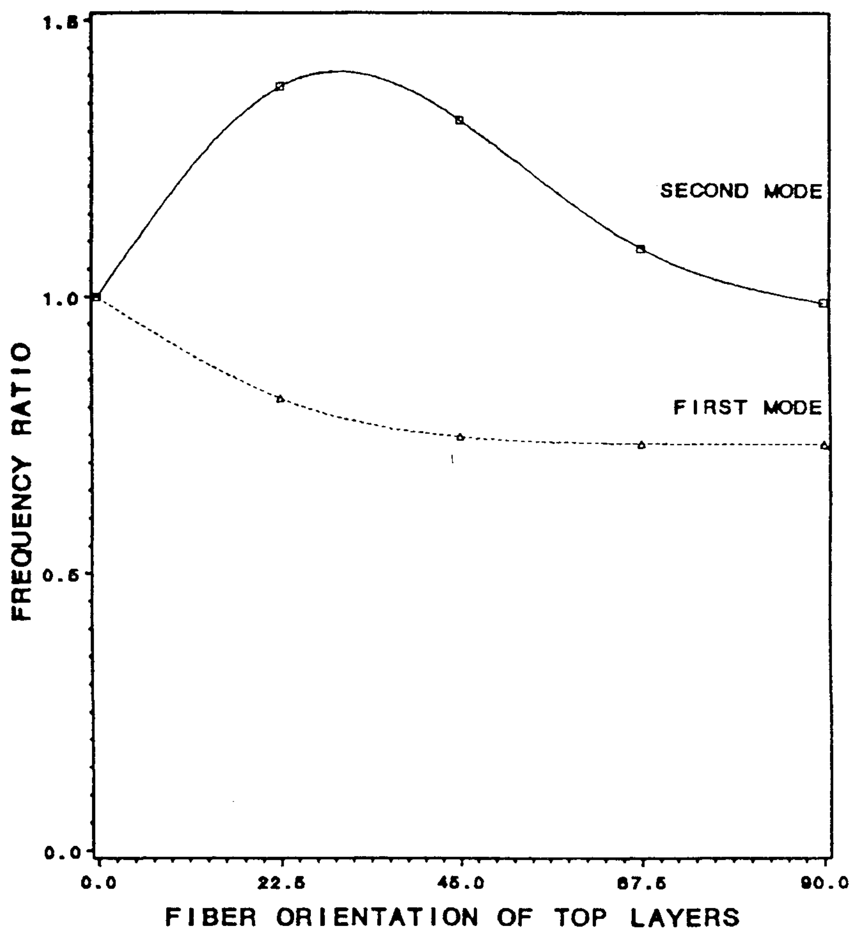

Fig. 3 Frequency variation for unsymmetrically laminated beams.

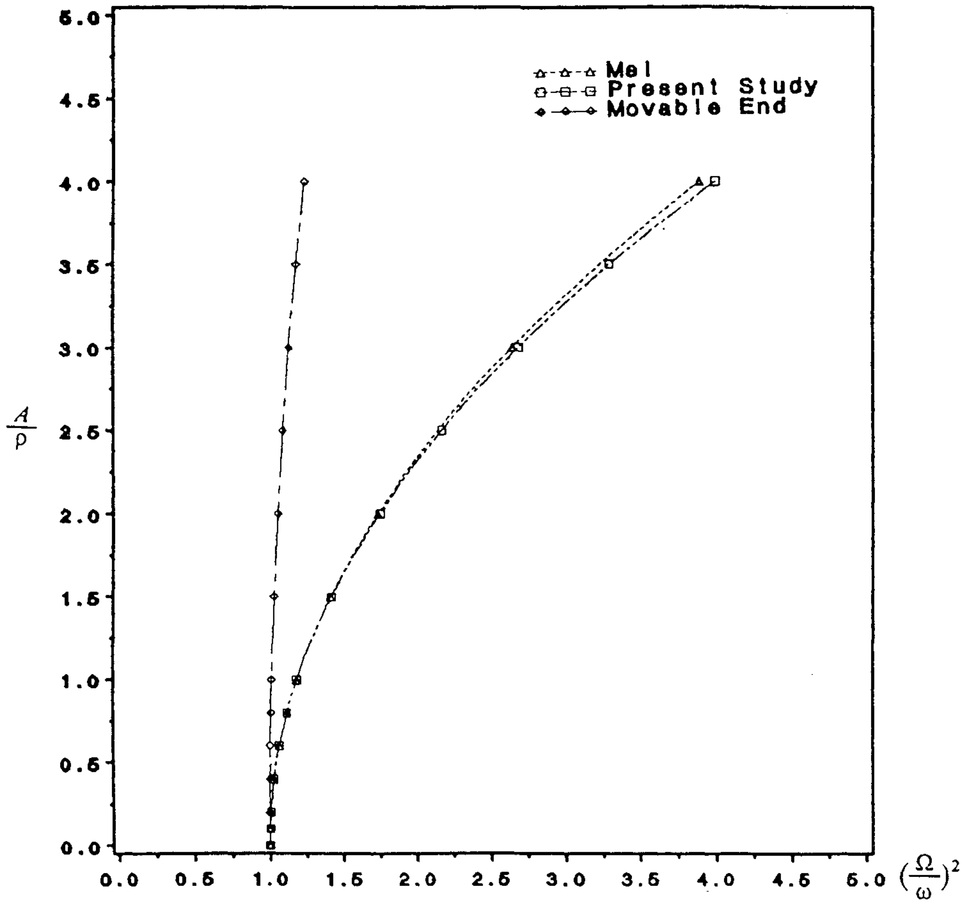

Fig. 4 Amplitude-frequency curve for a simply supported isotropic beam.

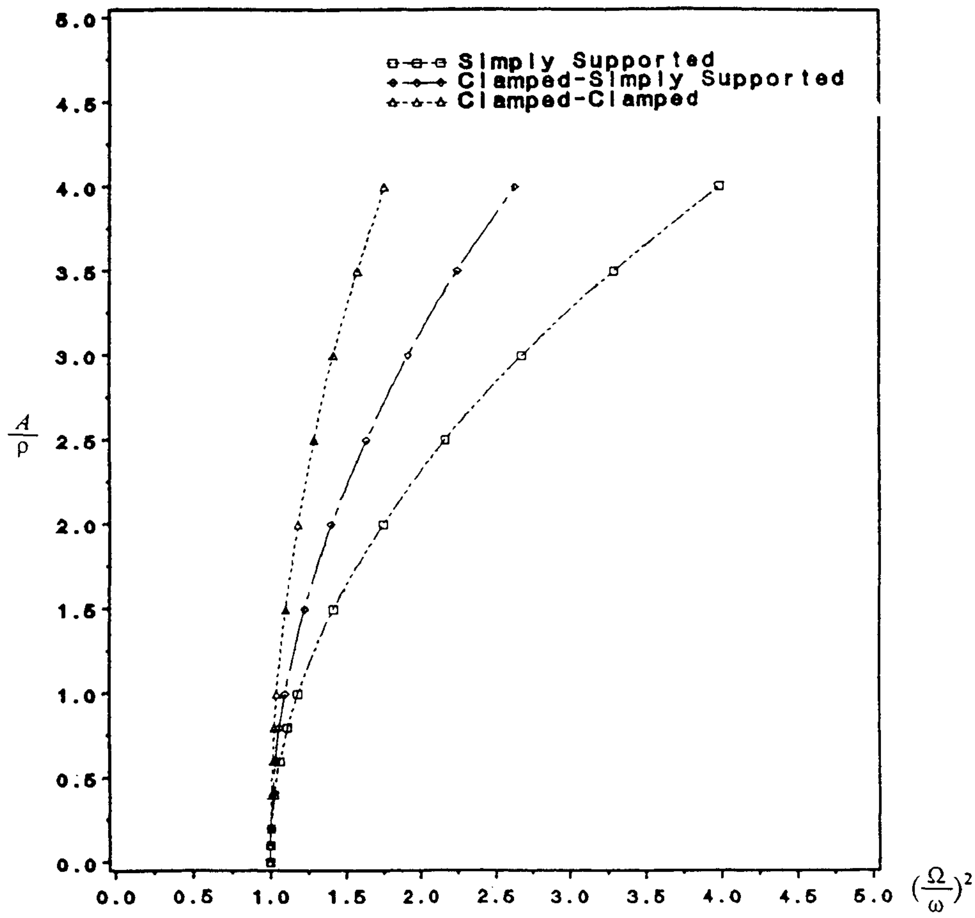

Fig. 5 Amplitude-frequency curve for isotropic beams with different support conditions.

bending only) w_b and θ_b appear at the bottom of the vectors $\{q\}$ and $\{\ddot{q}\}$, Eq. (25) may be rewritten as

$$\begin{bmatrix} k_{uu} & k_{uw} \\ k_{wu} & k_{ww} \end{bmatrix} \begin{Bmatrix} Q_u \\ Q_w \end{Bmatrix} + \begin{bmatrix} 0 & n_{uw_1} \\ n_{wu_1} & n_{ww_1} \end{bmatrix} \begin{Bmatrix} Q_u \\ Q_w \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & n_{ww_2} \end{bmatrix} \begin{Bmatrix} Q_u \\ Q_w \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & M_{ww} \end{bmatrix} \begin{Bmatrix} \ddot{Q}_u \\ \ddot{Q}_w \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{26}$$

where the subscript w refers to the transverse displacements due to bending, and the subscript u refers to the rest of the DOF. For the case of isotropic and symmetrically laminated beams, $[k_{uw}] = [k_{wu}] = 0$, since the coupling between the in-plane and the normal displacements is absent. Also, it may be noted that, for isotropic and symmetrically laminated beams, the submatrix $[n_{ww_1}]$ vanishes. The submatrices n_{uw_1}, n_{wu_1}, and n_{ww_1} are linear functions, and the submatrix n_{ww_2} is a quadratic function of the nodal displacements w_b and θ_b only.

Note that the variation of the mode is assumed to be the same as the linear mode. Let

$$\{Q_w\} = \{\bar{Q}_w\}\tau(t) \tag{27}$$

where $\{\bar{Q}_w\}$ is the normalized linear mode so that the maximum transverse displacement due to bending is unity, and $\tau(t)$ is a time function and will be written as τ in the following. Substituting Eq. (27) in Eq. (26), the in-plane displacement vector may be written as

$$\{Q_u\} = -[k_{uu}]^{-1}[k_{uw}]\{\bar{Q}_w\}\tau - [k_{uu}]^{-1}[\bar{n}_{uw_1}]\{\bar{Q}_w\}\tau^2 \tag{28}$$

Substituting Eq. (28) in Eq. (26), the equation of motion for normal displacements becomes

$$[k_{ww} - k_{wu}k_{uu}^{-1}k_{uw}]\{\bar{Q}_w\}\tau + [\bar{n}_{ww_1} - k_{wu}k_{uu}^{-1}\bar{n}_{uw_1} - \bar{n}_{wu_1}k_{uu}^{-1}k_{uw}]\{\bar{Q}_w\}\tau^2 + [\bar{n}_{ww_2} - \bar{n}_{wu_1}k_{uu}^{-1}\bar{n}_{uw_1}] \times \{\bar{Q}_w\}\tau^3 + [M_{ww}]\{\bar{Q}_w\}\ddot{\tau} = \{0\} \tag{29}$$

The bars over various submatrices indicate that these matrices have been evaluated using the normalized mode $\{\bar{Q}_w\}$. Equation (29) can be solved by multiplying all terms by $\{\bar{Q}_w\}^T$ to get an equation of the form

$$\ddot{\tau} + \alpha_1\tau + \alpha_2\tau^2 + \alpha_3\tau^3 = 0 \tag{30}$$

Solution of the Eq. (30) can be obtained using any of the available methods described by Nayfeh and Mook.[26]

If the function τ is expanded in a periodic solution of the form

$$\tau = \sum_{m=0}^{M} A_m \cos m\phi \tag{31}$$

where

$$\phi = \Omega t + \beta_0 \tag{32}$$

then Ω is given by[26]

$$\Omega = \omega\left\{1 + A_1^2\left[\frac{3}{4}\frac{\alpha_3}{\alpha_1} - \frac{5}{6}\left(\frac{\alpha_2}{\alpha_1}\right)^2\right]\right\}^{1/2} \tag{33}$$

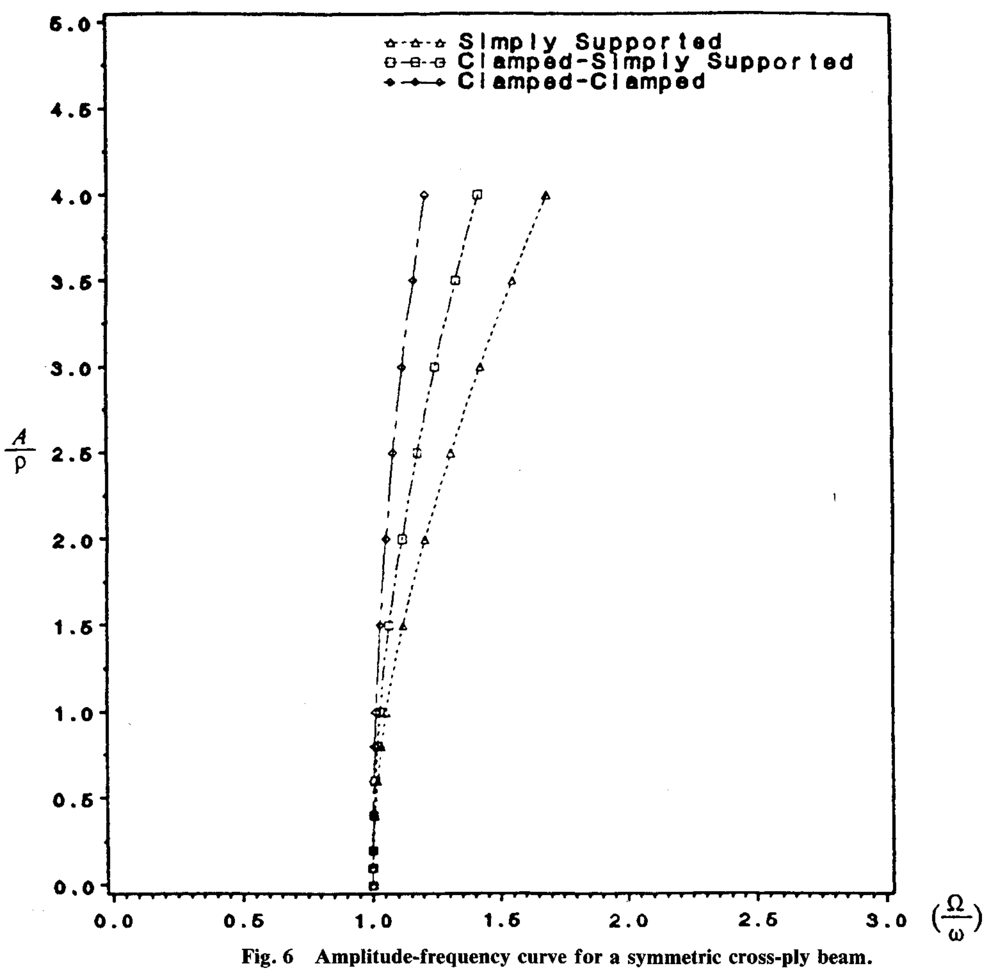

Fig. 6 Amplitude-frequency curve for a symmetric cross-ply beam.

The nonlinear period in the present study was obtained from Eq. (33). It is noted that this equation was also employed by Sinharay and Banerjee[27] for studying the large-amplitude free vibrations of shells of variable thicknesses.

Note that, for isotropic and symmetrically laminated beams, the coefficient of τ^2, α_2 is identically zero. The analysis of nonlinear vibrations for unsymmetrically laminated beams is thus significantly different from that of isotropic and symmetrically laminated beams. The bending-stretching coupling (the B matrix in the so-called A-B-D matrix) induces the quadratic term τ^2. As seen from Eq. (33), the effect of these terms is to reduce the nonlinear frequency. The effect of α_3 terms, in the present context, is in general to increase the frequency because, when vibrating with large amplitudes, the membrane forces caused by the $\frac{1}{2}(\mathrm{d}w_b/\mathrm{d}x)^2$ term are tensile. These membrane forces increase the bending stiffness of the beam.

Results

A number of examples were solved in order to evaluate the formulation, solution procedure, and computer program by comparing the obtained solutions with existing alternative solutions whenever possible. Note that all of the problems presented here were solved using the classical value for the shear correction factor $k = 5/6$, unless otherwise specified. The present element was tested by solving the static problems of isotropic and laminated cantilever beams under tip load and buckling of a simply supported beam. The buckling load was found to be significantly affected by the shear effects. The present results were found to be in good agreement with the previous results. The details of the examples are given in Ref. 28.

Free Vibrations of Graphite-Epoxy Cantilever Beams

Results for the free vibration of unidirectional fiber-reinforced beams were presented by Abarcar and Cunniff[14] in their study of the effect of fiber orientation on the mode of vibration of cantilever laminated beams. Their results showed that for certain fiber orientations, such as 30 deg, the interaction between bending and twisting can be quite strong, thus bringing into play "a new spectrum of natural frequencies which is different from the beam that vibrates in a pure bending fashion." This is why torsional mode shapes are present in the following results. Table 1 shows good agreement between the results obtained by this formulation and those obtained by Abarcar and Cunniff.[14]

Free Vibration of Unsymmetrically Laminated Panels and Beams

In order to evaluate the results for unsymmetrically laminated beams, the studies conducted by Thornton and Clary[29] and Thornton[30] on the vibration of composite material panels are referenced. These studies report data for the free vibration of square, unsymmetrically laminated cantilever panels. The panels were made of eight laminations of boron-epoxy. The four bottom laminations had an orientation of $\theta = 0$ deg whereas, for the four top plies, the fiber orientation varied from $\theta = 0$ deg (symmetric panel) to $\theta = 90$ deg. Results for the four different stacking sequences with increasing asymmetry show that the bending-stretching coupling effect reduces bending stiffness, thereby lowering vibration frequencies. The present results obtained with the beam element are within the approximate 10-20% range of discrepancy with the compared experimental results. This discrepancy is due to the modeling of a square panel as a one-dimensional beam element. The results are shown in Table 2.

Additional results are presented for unsymmetrically laminated beams with length-to-width ratio of 5:1. Similar patterns exhibited by the square panels characterize the vibrations of the beams. Natural frequencies decrease with increasing asymmetry of the laminate, as shown in Table 3. Figure 3 shows the plot of the ratio of the beams' natural frequencies (first two modes) to the frequency of a symmetric beam $[0_4/0_4]$. It is noted that similar trends were observed for the square plate by Thornton and Clary.[29]

Table 1 Natural frequencies of a 30-deg graphite-epoxy cantilever beam

Mode no.[a]	4 elements f_n, Hz	Predicted[14] f_n, Hz	Experimental[14] f_n, Hz
1-B	52.65	52.7	52.7
2-B	329.78	329.3	331.8
3-B	928.29	915.9	924.7
1-T	1818.42	1896.5	1827.4

[a] B = bending mode, T = torsion mode.

Table 2 Natural frequencies of an unsymmetric boron-epoxy plate $[45_4/0_4]$

Mode no.	4 elements f_n, Hz	Finite-element[30] f_n, Hz	Experimental[30] f_n, Hz
1	30.20	24.3	24.4
2	36.64	48.5	55.1
3	104.20	118.0	141.0
4	172.80	151.0	157.0
5	201.67	193.0	208.0

Table 3 Natural frequencies of unsymmetrically laminated beams

Mode no.	$\theta = 0$ deg f_n, Hz	$\theta = 22.5$ deg f_n, Hz	$\theta = 45$ deg f_n, Hz	$\theta = 67.5$ deg f_n, Hz	$\theta = 90$ deg f_n, Hz
1	107.36	87.71	80.30	78.96	78.87
2	340.05	470.40	449.22	370.71	337.61
3	673.58	560.64	505.77	495.40	494.83
4	1020.17	1349.97	1331.82	1112.12	1031.44

Nonlinear Vibrations of Isotropic Beams

Results for the nonlinear vibrations are first presented for the case of isotropic beams. Nonlinear vibrations of isotropic beams have been studied by Mei[31] and Ray and Bert.[32] The plot of the amplitude-radius of gyration vs the nonlinear-linear frequency ratio square $(\Omega/\omega)^2$ is referred to as the amplitude-frequency curve. Figure 4 shows the amplitude-frequency curve for a simply supported beam with immovable ends. The results obtained by the present formulation are compared with those obtained by Mei[31]; the agreement is good. Also, results for the simply supported beam with movable end (one end is allowed to move in the x direction) are given for comparison. Note that, for the movable end case, the problem reduces to that of linear vibrations. The amplitude-frequency curves for three different support conditions are shown in Fig. 5.

Nonlinear Vibrations of Symmetrically Laminated Beams

Results for the nonlinear vibrations of a cross-ply $[0/90]_s$ beam and an angle ply $[45/-45]_s$ are shown in Figs. 6 and 7, respectively. The material properties used were those of graphite-epoxy: $E_1 = 18.5 \times 10^6$ psi, $E_2 = 1.6 \times 10^6$ psi, $\upsilon_{12} = 0.25$, $G_{12} = 0.65 \times 10^6$ psi, and $\rho = 0.055$ lb/in^3. These results were obtained using six elements. Note that for this plot the nonlinear-linear frequency ratio is not squared. The nonlinearity is of a hardening type, and the effect of the clamped support is to reduce the nonlinearity. This behavior was also observed for the isotropic beam case. Also, note that the effect of having the fibers running neither parallel nor perpendicular to the longitudinal axis of the beam is to increase the nonlinearity as compared to the cross-ply beam.

Nonlinear Vibrations of Unsymmetrically Laminated Beams

Results for a quasi-isotropic $[0/45/-45/90]_T$ and for a $[0/30/-30/90]_T$ unsymmetrically laminated beams are pre-

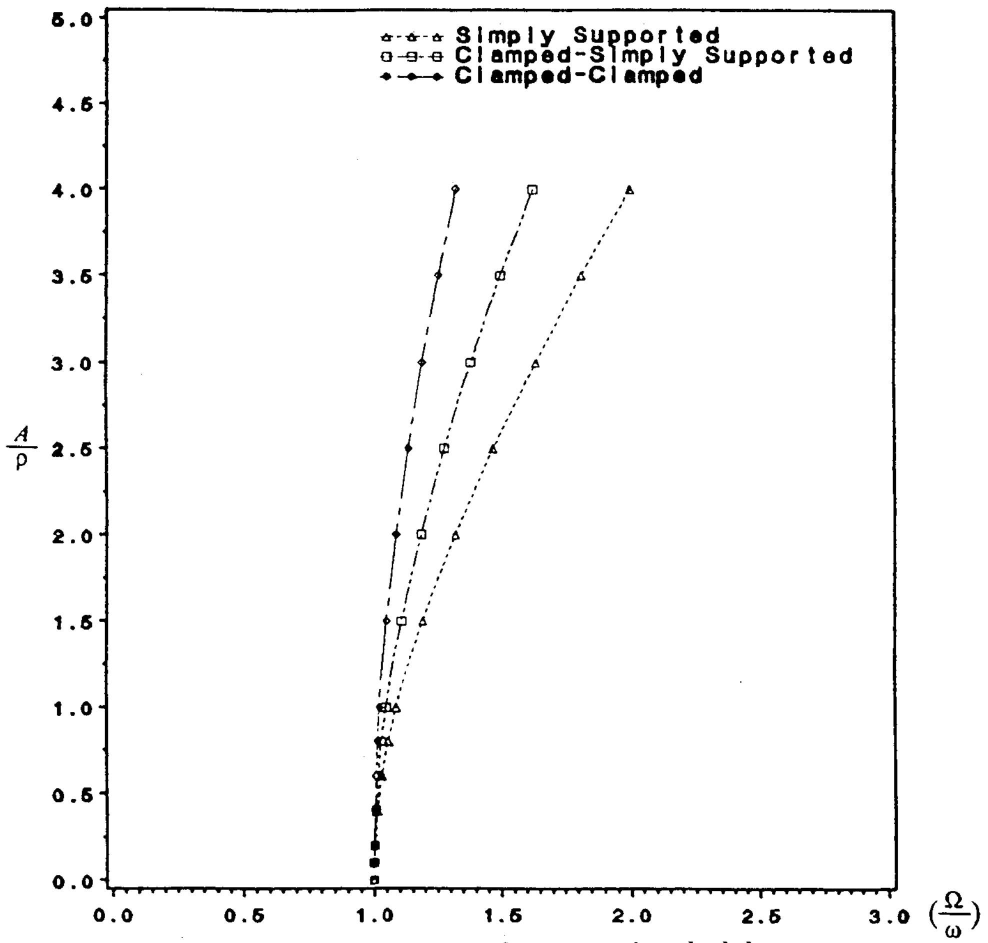

Fig. 7 Amplitude-frequency curve for a symmetric angle-ply beam.

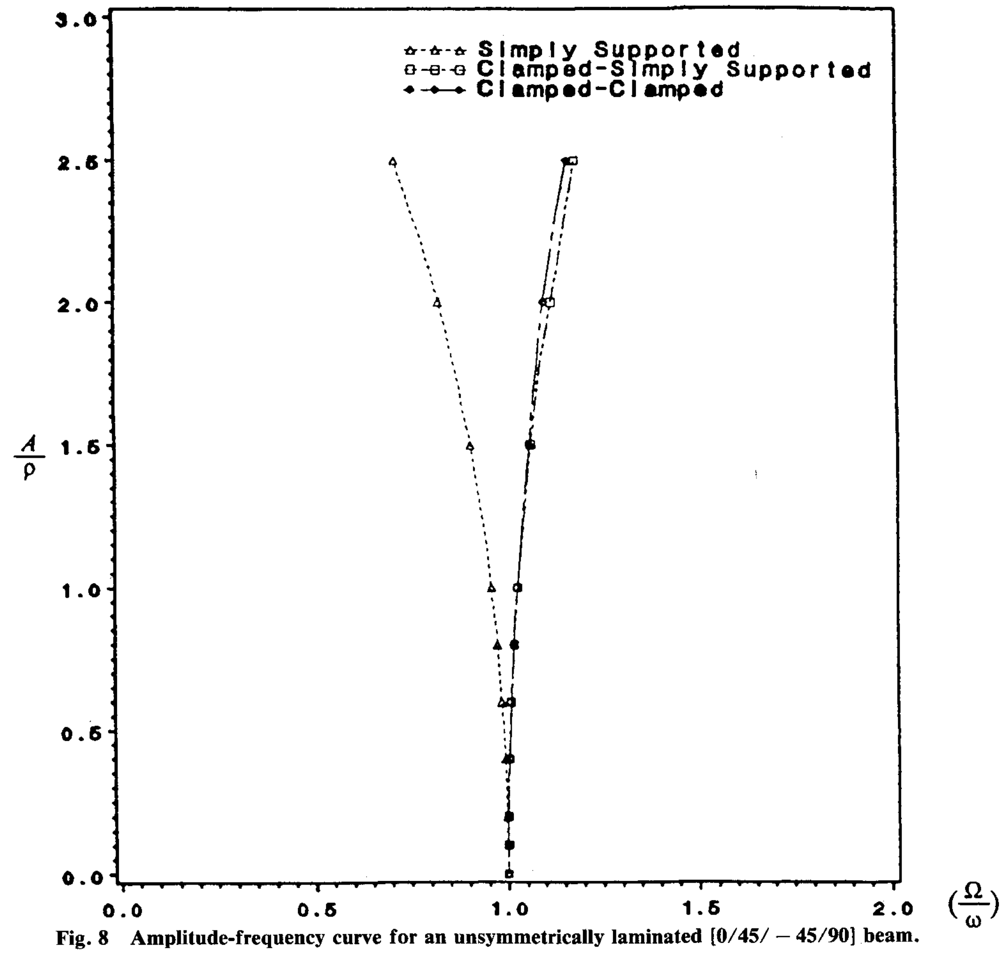

Fig. 8 Amplitude-frequency curve for an unsymmetrically laminated [0/45/ − 45/90] beam.

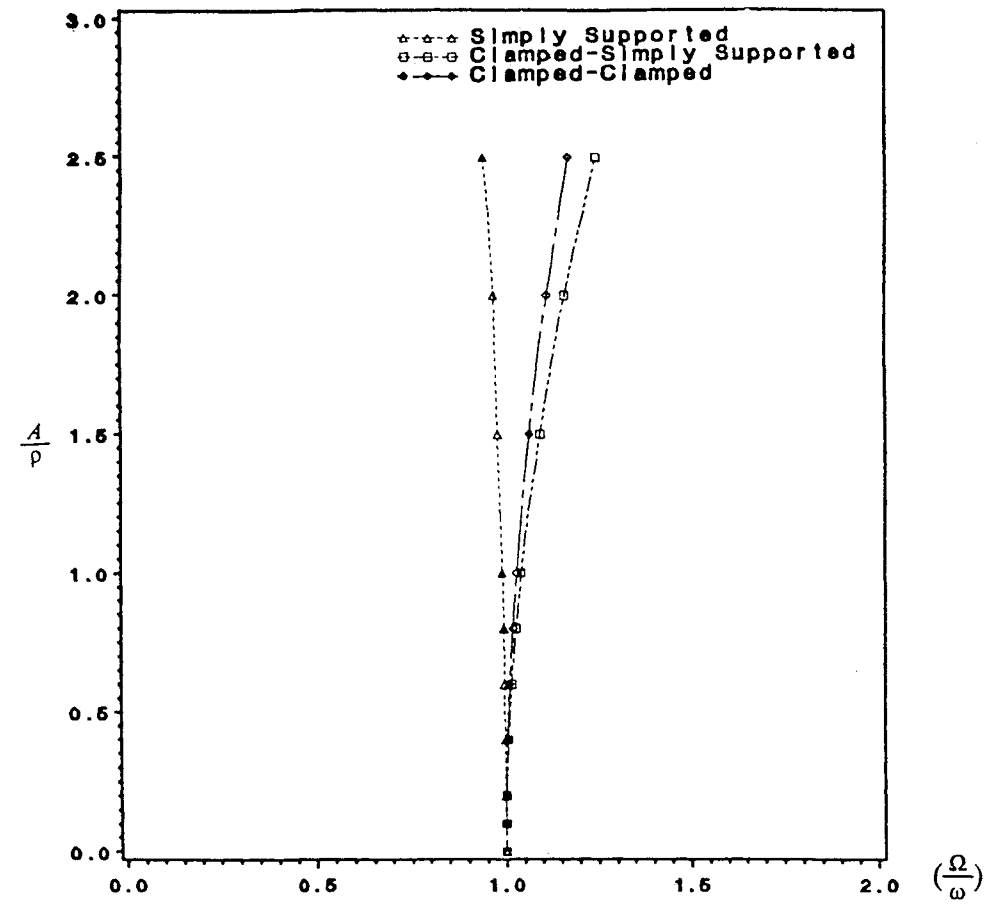

Fig. 9 Amplitude-frequency curve for an unsymmetrically laminated [0/30/ − 30/90] beam.

sented here. The material properties used were the same as those given earlier for the symmetrically laminated beams. As previously mentioned, the effect of the unsymmetry is to introduce a nonzero coefficient of τ^2 whereas, for all of the previous nonlinear examples, the coefficient of τ^2 was found to be zero. The results are shown in Figs. 8 and 9. For the simply supported beam, the effect of τ^2 is to change the nonlinearity to a softening type. For the clamped-simply supported and clamped-clamped beams, the effect of τ^2 is to reduce the nonlinearity. Also, note that results are given for amplitude-radius of gyration ratios up to 2.5. At higher amplitudes, the single-mode assumption may lead to inaccurate results in the analysis of unsymmetrically laminated beams. The bending-stretching effect may lead to large variations in the mode shapes. In order to evaluate the nonlinear response at higher amplitudes, a multimode analysis would be needed.

Concluding Remarks

The formulation and computer program developed in this study are tools for analyzing the static, free-vibration, buckling, and nonlinear vibration behavior of laminated beams.

The present formulation allows for the analysis of unsymmetrically laminated beams, which are characterized by bending-stretching coupling. The beam element has 20 degrees of freedom, which allows modeling of the different couplings that characterize laminated composite beams. Also, the present formulation takes into account the effect of shear deformation, which can be of considerable importance in the analysis of composite materials characterized by a very low transverse shear moduli as compared to the in-plane tensile moduli.

The bending-stretching effect on the nonlinear vibrations of unsymmetrically laminated beams was found to change the nonlinearity from a hardening type to a softening type, for certain boundary conditions, as opposed to a hard-spring behavior observed in isotropic and symmetrically laminated beams. Also, the in-plane boundary conditions were found to have a significant effect on nonlinear responses.

There are areas for additional work. A different approach for including shear deformation would be more appropriate than the present approximation using shear factors. A possible approach would be the one similar to that of Hinrichsen and Palazotto[33] used for thick laminated plates. An alternative approach is the one developed by Stein and Jegley.[34]

The shear deformation has been neglected only in nonlinear vibrations. Inclusion of shear deformation in the nonlinear vibrations may be important for analyzing the nonlinear response of thick laminated beams. In a recent study, Mei and Prasad[35] pointed out that the effect of transverse shear may be quite significant for plate lengths less than 20 times the thickness. However, for moderately thick plates, small deflection theory with shear deformation yields accurate predictions of maximum deflection, frequency, and stresses. For thin plates (length/thickness > 50), the large deflection theory with transverse shear effects neglected should give reasonable results. Also, it is believed that a multimode analysis would be more appropriate for studying the nonlinear behavior of unsymmetrically laminated beams at high amplitudes.

References

[1]Fricker, A. J., "A Simple Method for Including Shear Deformation in Thin Plate Elements," *International Journal for Numerical Methods in Engineering,* Vol. 23, July 1986, pp. 1355–1366.

[2]Di Sciuva, M., "Bending, Vibration and Buckling of Simply Supported Thick Multilayered Orthotropic Plates: An Evaluation of a New Displacement Model," *Journal of Sound and Vibration,* Vol. 105, March 1986, pp. 425–442.

[3]Whitney, J. M. and Pagano, N. J., "Shear Deformation in Heterogeneous Anisotropic Plates," *Journal of Applied Mechanics,* Dec. 1970, pp. 1031–1036.

[4]Reddy, J. N. and Kuppusamy, T., "Natural Vibrations of Laminated Anisotropic Plates," *Journal of Sound and Vibration,* Vol. 94, May 1984, pp. 63–69.

[5]Bhashyam, G. R. and Gallagher, R. H., "An Approach to the Inclusion of Transverse Shear Deformation in Finite Element Plate Bending Analysis," *Computers and Structures,* Vol. 19, Jan.–Feb. 1984, pp. 35–40.

[6]Phan, N. D. and Reddy, J. N., "Analysis of Laminated Composite Plates Using a Higher-Order Shear Deformation Theory," *International Journal for Numerical Methods in Engineering,* Vol. 21, Dec. 1985, pp. 2201–2219.

[7]Reddy, J. N., "A Refined Nonlinear Theory of Plates with Transverse Shear Deformation," *International Journal of Solids and Structures,* Vol. 20, Sept.–Oct. 1984, pp. 881–896.

[8]Chen, A. T. and Yang, T. Y., "Static and Dynamic Formulation of a Symmetrically Laminated Beam Finite Element for a Microcomputer," *Journal of Composite Materials,* Vol. 19, Sept. 1985, pp. 459–475.

[9]Teh, K. K. and Huang, C. C., "The Vibrations of Generally Orthotropic Beams, A Finite Element Approach," *Journal of Sound and Vibration,* Vol. 62, Jan. 1979, pp. 195–206.

[10]Whitney, J. M., Browing, C. E., and Mair, A., "Analysis of the Flexure Test for Laminated Composite Materials," American Society for Testing and Materials, Philadelphia, PA, STP-546, 1974, pp. 30–45.

[11]Teoh, L. S. and Huang, C. C., "The Vibration of Beams of Fiber Reinforced Material," *Journal of Sound and Vibration,* Vol. 51, April 1977, pp. 467–473.

[12]Murty, A. V. K. and Shimpi, R. P., "Vibration of Laminated Beams," *Journal of Sound and Vibration,* Vol. 36, Sept. 1974, pp. 273–284.

[13]Miller, A. K. and Adams, D. F., "An Analytic Means of Determining the Flexural and Torsional Resonant Frequencies of Generally Orthotropic Beams," *Journal of Sound and Vibration,* Vol. 41, Aug. 1981, pp. 433–449.

[14]Abarcar, R. B. and Cunniff, P. F., "The Vibration of Cantilever Beams of Fiber Reinforced Material," *Journal of Composite Materials,* Vol. 6, Oct. 1972, pp. 504–517.

[15]Hu, M. Z., Kolsky, H., and Pipkin, A. C., "Bending Theory for Fiber Reinforced Beams," *Journal of Composite Materials,* Vol. 19, May 1985, pp. 235–249.

[16]Reddy, J. N., "Geometrically Nonlinear Transient Analysis of Laminated Composite Plates," *AIAA Journal,* Vol. 21, April 1983, pp. 621–629.

[17]Reddy, J. N. and Chao, W. C., "A Comparison of Closed-Form and Finite Element Solutions of Thick Laminated Anisotropic Rectangular Plates," *Nuclear Engineering and Design,* Vol. 64, April 1981, pp. 153–167.

[18]Elishakoff, I. and Lubliner, E., "Random Vibration of a Structure Via Classical and Nonclassical Theories," *IUTAM Symposium on Probabilistic Methods in the Mechanics of Solids and Structures,* edited by S. Eggewertz and N. C. Lind, Springer-Verlag, Berlin, 1984, pp. 455–465.

[19]Thambiratanam, D. P., "Transient Response of Beam Under Initial Stress," *Journal of Engineering Mechanics,* Vol. 110, Oct. 1984, p. 1544.

[20]Cowper, G. R., "The Shear Coefficient in Timoshenko's Beam Theory," *Journal of Applied Mechanics,* June 1966, pp. 335–339.

[21]Bhat, S. U. and de Oliveira, J. G., "A Formulation for the Shear Coefficient of Thin-Walled Prismatic Beams," *Journal of Ship Research,* Vol. 29, Jan. 1985, pp. 51–58.

[22]Bert, C. W., "Simplified Analysis of Static Shear Factors for Beams of Nonhomogeneous Cross Section," *Journal of Composite Materials,* Vol. 7, Oct. 1973, pp. 525–529.

[23]Bert, C. W. and Gordaninejad, F. "Transverse Shear Effects in Bimodular Composite Laminates," *Journal of Composite Materials,* Vol. 17, July 1983, pp. 282–291.

[24]Sathyamoorthy, M., "Nonlinear Analysis of Beams Part I: A Survey of Recent Advances," *Shock and Vibration Digest,* Vol. 14, Aug. 1982, p. 19–35.

[25]Sathyamoorthy, M., "Nonlinear Analysis of Beams Part II: Finite Element Methods," *Shock and Vibration Digest,* Vol. 14, 1982, pp. 7–18.

[26]Nayfeh, A. H. and Mook, D. T., *Nonlinear Oscillations,* Wiley, New York, Sept. 1979.

[27]Sinharay, G. C. and Banerjee, B., "Large Amplitude Free Vibrations of Shells of Variable Thickness," *AIAA Journal,* Vol. 24, June 1986, pp. 998–1004.

[28]Raciti, S. and Kapania, R. K., "A Twenty DOF Element for Nonlinear Analysis of Unsymmetrically Laminated Beams," Center for Composite Materials and Structures, Virginia Polytechnic Inst. and State Univ., Blacksburg, VA, CCMS-87-03, March 1987.

[29]Thornton, E. A. and Clary, R. R., "A Correlation Study of Finite Element Modeling for Vibrations of Composite Material Panels," *Composite Materials: Testing and Design (Third Conference),* ATM STP 546, 1974, pp. 111–129.

[30]Thornton, E. A., "Free Vibrations of Unsymmetrically Cantilevered Composite Panels," *Shock and Vibration Bulletin,* Pt. 2, Aug. 1976, pp. 79–88.

[31]Mei, C., "Nonlinear Vibration of Beams by Matrix Displacement Method," *AIAA Journal,* Vol. 10, Aug. 1972, pp. 355–357.

[32]Ray, J. D. and Bert, C. W., "Nonlinear Vibrations of a Beam with Pinned Ends," *Journal of Engineering for Industry,* Vol. 91, Ser. B, Nov. 1969, pp. 997–1004.

[33]Hinrichsen, R. L. and Palazotto, A. N., "Nonlinear Finite Element Analysis of Thick Composite Plates Using Cubic Spline Functions," *AIAA Journal,* Vol. 24, Nov. 1986, pp. 1836–1842.

[34]Stein, M. and Jegley, D. C., "Effects of Transverse Shearing on Cylindrical Bending, Vibration, and Buckling of Laminated Plates," *AIAA Journal,* Vol. 25, Jan. 1987, pp. 123–129.

[35]Mei, C. and Prasad, C. B., "Effects of Large Deflection and Transverse Shear on Response of Rectangular Symmetric Composite Laminates Subjected to Acoustic Excitation," AIAA Paper 87-0933, April 1987.